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REVIEWS

Problems and Methods of Network Control

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Abstract—Control of network systems, or network control, is a rapidly developing field of modern automated control theory. Network control is characterized by a combination of the classical control theory toolbox (linear systems, nonlinear control, robust control and so on) and conceptually new mathematical ideas that come primarily from graph theory. Methods of network control let one solve analysis and synthesis problems for complex systems that arise in physics, biology, economics, sociology, and engineering sciences. In this survey, we present the main fields of application for modern theory of network control and formulate its key results obtained over the last decade.

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1. INTRODUCTION

The wide spread of network systems, understood as a collection of subsystems (nodes) linked by physical or informational connections, has led to a number of novel control theory problems. Numerous examples of network systems include multiprocessor systems for information transmission and processing, transportation and logistical networks, high-tech industrial networks, coordinated control systems for group motion of aerial and underwater vehicles and mobile robots, distributed control systems for electric circuits, complex crystal lattices and nanostructural objects, neural networks, networks of genetic and biochemical oscillators, large social groups, and biological formations. One sure sign of the relevance and scientific importance of the problem is the booming research literature on complex networks and network systems. Surveys, reports, and special issues of journals have appeared [1–8]. Conferences and seminars have been specifically devoted to control in network systems [9]. A keyword search in the *Web of Science* database shows that the number of papers in refereed journals on this topic has doubled over the last 5–6 years. Search for the keyword "networks" in the proceedings of the largest control theory conference 2013 IEEE Conference on Decision and Control (CDC 2013) yields more than 500 papers out of 1270. Finally, four out of ten most cited papers from the oldest and most prestigious journal in the field of automatic control, *IEEE Transactions on Automatic Control*, are papers on network control that have appeared only about 10 years ago (we will mention these papers below). All of this evidences a rapid growth and already achieved high level of relevance for the topic of control in network systems (see Fig. 1).

Almost 50 years ago, a prominent Russian researcher Ya.Z. Tsypkin in his wonderful book "Adaptation and Control in Automatic Systems" [10] announced the advent of a new, third period in the development of control theory, namely the adaptive period when models and methods of adaptive control are in the center of attention. The "adaptive" period lasted for almost 30 years, nearly till the end of the XX century. By now, we can speak of the arrival of the fourth, network period.

Existing literature on network control systems can be roughly divided into two large classes. The first class is *networked control*, and the second is control *of networks* or *in networks*. The

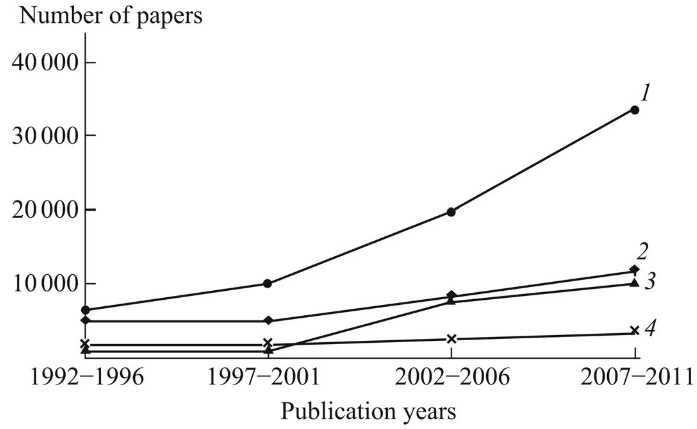

Fig. 1. Publication dynamics in main subfields of control theory in *Web of Science* indexed journals: (*1*) control in networks (search for keywords "control and network"), (*2*) adaptive control (search for keywords "control and adapt"), (*3*) intelligent control systems (search for keywords "control and intelligen"), (*4*) robust control (search for keywords "control and robust").

first direction of study is primarily related to control under communication and computational constraints that are present in virtually any complex computer controlled system. Sample problems considered in this field are the influence exerted on control systems by quantization and sampling, delays and data losses or dropouts, bounded data rate, or channel capacity. These kind of problems go beyond the scope of this survey, and main problems and most important results of the "networked control" theory have been surveyed in recently published works [11–13].

In this survey, we consider the main problems and applications of the second subfield of network control theory which studies the problems of control in networks. Specific problems in this field are often denoted by such terms as "group control," "cooperative control," "multiagent control." Network control systems differ from classical in both the structure of the control object and the structure of "controllers," or control algorithms, which in network control theory are also often called *protocols*. A control object is subdivided (either naturally or artificially) into separate subsystems (nodes) which are usually not controlled from a single center but make and implement decisions independently based on the information available to them. This kind of behavior is called *agent* behavior, and respectively the nodes/subsystems are often called *agents*, and the entire system is called *multiagent*.[1] A way to represent a complex system as a group of interacting agents is often called *agent-based modeling*. Hence a control algorithm in a network (multiagent) system must be *distributed* and *decentralized*. The former term means that each node (agent) is controlled by its own independent controller, and the second term assumes that such controllers only use "local" information concerning the system (which usually means the state of the node itself and several "neighboring" nodes).

A wide class of network multiagent systems is defined in continuous time by mathematical models such as

$$\dot{x}_i = F_i(t, x_i, u_i) + \sum_{j=1}^{N} \alpha_{ij}(t)\varphi_{ij}(x_i, x_j), \quad y_i = h(x_i, u_i), \quad i = 1, \ldots, N, \tag{1}$$

[1] The term "multiagent control" most often appears in the situation when nodes of the network are completely autonomous from the start, and information connections between them arise only as they apply a joint control algorithm which is intended to achieve a certain common, or *cooperative* goal (examples include consensus algorithms for independent agents that we consider below and numerous coordination problems for autonomous mobile robots).

where N denotes the number of nodes in the network, $x_i(t)$ are state vectors for the nodes, $u_i(t)$ are the inputs (controls), $y_i(t)$ are the measured variables (outputs), functions $F_i(\cdot)$ characterize the agents' own (local) dynamics, functions $\varphi_{ij}(\cdot)$ characterize the interactions between agents, and the numbers α_{ij} define a (weighted) graph of system connections. This graph, similar to the dynamics of the nodes themselves, may be nonstationary (in particular, connections between agents may appear and disappear again). Coupling functions φ_{ij} and gains α_{ij} may be fixed in advance, and in this case we study the properties of system (1) as a dynamical system with inputs u_i and outputs y_i, or can be a part of a distributed control algorithm.

Analysis and synthesis problems for such systems have been considered in a large number of works; see, e.g., [1–8, 14–24]. The adequate mathematical formalism here is a mix of stability theory (usually Lyapunov or input-output methods) and graph theory (the properties of a system significantly depend on the spectrum of the so-called Laplacian matrix of the graph of connections defined via its adjacency matrix). The idea of theoretical works in this direction is to formulate dynamical properties of the system (1) that correspond to a specific type of its behavior (either observed in nature or preferable for the system's designer) and then establish conditions to achieve these properties. Here for natural systems (physical, biological, and so on) the rules of interaction between agents are defined by the actual physical laws, and we solve analysis problems. For technical systems, in addition to that we solve synthesis problems for rules (algorithms) of interaction that ensure achieving given properties, i.e., given properties work as control objectives.

One of the main control objectives is **synchronization**, which means that agents act in a coherent fashion. For instance, complete or partial coordinate synchronization means that agent states or their observed outputs become asymptotically closer to each other: $\|x_i(t) - x_j(t)\| \to 0$ or $\|y_i(t) - y_j(t)\| \to 0$ as $t \to \infty$. Below we will give a general definition of synchronization that encompasses its various forms occurring in nature and in technology [25, 26]. To achieve this goal, the dynamical system should possess the properties of **partial stability, or stability with respect to a function**. This notion was introduced already by A.M. Lyapunov, but it was systematically studied only in the second half of the XX century, starting from the works of V.V. Rumyantsev and his successors. A special case of synchronization problems are problems of achieving **consensus** [3, 4, 14, 15, 27, 28], where states or outputs of the agents must converge to a common value (e.g., to the average of initial states) or a general predefined trajectory.[2] **Group control** problems (**formation** control) [2, 23] also reduce to partial coordinate synchronization problems with the virtual leader method [16]. The partial stability property is similar to the notion of **set stability**, which is a convenient way to describe the properties of achieving control objectives under disturbances.

For large numbers of agents (which in some problems is in the thousands or even millions), the requirement of having all agents to exhibit desired behavior may turn out to be too strict. In such cases, one chooses a characteristic point in the set of agent states (center, leader, center of mass), and desired behavior is defined as a given behavior of the center given that all agents deviate from this behavior in a bounded way. For such types of behavior, researchers introduce notions borrowed from biology, e.g., **swarming** or **flocking**, and establish conditions under which the system begins to exhibit the corresponding properties [17–19]. Finally, a number of works have been devoted to the **rendezvous** problem, where the objective is for all agents to meet in a given place at a given time [20–22].

In subsequent sections, we show mathematical settings of some basic problems and formulations of the corresponding results in the above-mentioned fields. We show approaches to systematizations of known results [23–25]. We also pay special attention to adaptive control problems in networks and

[2] As we will explain in detail below, this terminology cannot be considered established. For instance, in many works the term "consensus" is used as a synonym of coordinate synchronization.

approaches based on the passification method [29–34]. But before we proceed to formal exposition, we consider a few examples of practical problems that lie in the field of network control.

2. NETWORK CONTROL PROBLEMS

Industrial and economic networks. One of the best known and widely considered classes of dynamical networks are industrial and economic networks that include financial and transportation networks. Modern transnational industrial systems (corporations) have global geographic location. Raw material is mined in some countries and on some continents, individual units are manufactured in others, assembly is done in the third, and clients who receive the ready product may be located in some fourth countries. At the same time, the production process in each manufacturing node of the network must proceed smoothly without problems with either raw materials or required units, the level of automation in each node must ensure information exchange and the interaction of measurement and actuating devices (sensors and actuators) via wired and wireless communication channels in order to ensure stability and accuracy of operation for all equipment. Besides, warehouses in each node must store sufficient but not excessive reserves of raw material and units, and bank accounts in each node must have sufficient but not excessive capital reserves to ensure the entire production process. The ready product must be timely delivered to the customers via transportation networks, and payment for the product must be timely received at the bank accounts of industrial nodes via financial networks. Formalization of control problems described above is not too hard, but solving such problems is complicated because mathematical models of manufacturing and exchange processes are non-uniform, and the dimensions of the resulting data points are large (in the hundreds and thousands) or even gigantic (in the millions and billions). Nevertheless, there appear a large number of works on the control over industrial networks, and interest to these problems continues to grow [35–37]. These problem setings are also closely related to dynamic network models of markets, auctions, and economies that define the interaction of trade and economic agents [38–41].

Group control over land, air, and marine vehicles. Networks of mobile robots, unmanned aerial vehicles, surface and underwater vehicles have attracted a lot of attention for more than a decade [14, 22, 27, 42–46]. Such networks can solve numerous monitoring and search problems over a given territory or aquatory, online photo and video surveillance and so on. The recent boom of construction and application of unmanned aerial vehicles (UAV) has opened up new possibilities: joint carrying of loads, construction of buildings by collectives of robots, self-assembling of constructions in the air and so on. Control over such networks requires one to develop control methods for networks of dynamical systems with control objectives corresponding to various behavior types of a "collective" of robots. Excellent results in group control over a group of quadrocopters have been demonstrated in a plenary talk of Raffaello d'Andrea (ETH, Zurich) on the CDC-2013 conference [47]; see also [48, 49].

Problems of cooperative robot behavior are especially interesting for nonstandard control objectives. One example of this class of problems is robot soccer, where the control objective is to score a goal under adversarial actions of the opposite team. This leads to a number of complex intermediate problems: encircling moving obstacles, planning a collection of player trajectories, implementing maneuvers, and so on. Over the latest years, serious progress has been achieved in solving problems of this kind; a survey of general approaches and results can be found, for instance, in [50]. A similar class of air traffic control problems is becoming further complicated by the increase in "population density" of the air space, especially near large cities and airports.

Power networks. An important class of network systems are electric power networks that consist of a large number of power generators some of which are connected with power lines. The main control objective here is to achieve and support stable synchronous operation of generators under changing load and various disturbances. If one achieves reliable synchronization, one can reduce the number of emergency situations (blackouts) and reduce various kinds of losses. A large number of generators and consumers in the network, problems of measuring the corresponding variables such as the phase (instantaneous value of voltage or current), and wide ranges of network parameters present significant obstacles to the control of power networks.

Another obstacle that complicates control problems for power networks is their complex dynamics. Apart from nonlinear dynamics of the generators, one has to take into account the large-scale structure and complex topology of the network. This leads to a substantially nonlinear character of network behavior defined in terms of full or partial stability, bifurcations, and chaos. Moreover, sufficiently detailed models of real life power networks are defined by a collection of a large number of differential and algebraic equations, which further complicates their study and control over them [51, 52].

New possibilities in the control of electric power networks have arisen with the development and production of such devices as PMU (Phase Measuring Units) and WAMS (Wide Area Measurement System) that ensure precise measurements of time and phase variables of the network, and other flexible systems of transmitting alternate current, in particular FACTS (Flexible Alternate Current Transmission Systems). These and other factors have motivated the concept of a "smart grid" [53–56], which in Russian publications is sometimes called an "active-adaptive network" or "intelligent network" [57].

This concept was born in the U.S. and European Union countries, where it has served as a foundation for the national policy of energy and innovation development. The reasons for the new concept to arise, both in the whole world and in Russia, are related to a number of factors: technological progress (increasing the automation level, development of new technologies, rapid growth of the number of small generating energy sources); a growth in the consumers' demands (requirements on the range and quality of services, decreasing prices); reduction in reliability (due to increased wear and tear in the equipment, deterioration in the electric supply reliability, high level of losses in transforming, transmitting, distributing, and using the energy); increased requirements to energy efficiency and ecological safety, and so on. A number of control problems for power networks with methods of adaptive and robust control have been considered in recent works [58–61].

Distributed communication systems and computer networks. In communication theory, the study of distributed networks began long before "network" control theory ever arose [62, 63]. A distributed communication network consists of a number of receiving and transmitting nodes connected by a nontrivial topology of heterogeneous communication lines (either wired or wireless). Examples may include telephone and cable networks, mobile communication networks, or any computer network. A number of problems solved in the theory of distributed communication networks are immediately related to control over multiagent and network systems. One such problem is the problem of *time synchronization*, i.e., phase and frequency synchronization of tick generators. Despite the fact that distributed algorithms for automated tuning of time in distributed networks have been known for a long time (see, e.g., the survey [64]), tools for their strict mathematical analysis have appeared only very recently [65, 66] in the form of studying *consensus algorithms* that are considered below. Another classical problem is the *dynamic load balancing* problem that arises in any large communication network: in the presence of several possible paths of delivering a packet from point A to point B (with, generally speaking, different throughput), distribute traffic in real time between these paths to minimize the average delivery time. A similar problem can be posed for a system of several servers that process requests or a cluster of several processors [67]: how do we redistribute the load between servers/processors in such a way that the average time of processing a request is as small as possible. Load balancing algorithms based on *local voting* *protocols* (which are, in essence, synchronization and consensus algorithms) have been studied in recent works [68, 69].

Ecological networks. Ecological systems are characterized by the fact that they are spatially distributed and reflect the interaction of multiple species and populations, which leads to a complex character of their corresponding networks. Connections between species are generated by trophic (food) chains that define "who eats who" ("food web") [70]. Control over ecological networks can be intended to preserve stable existence of populations, defend them from extinction, keep population size of a species inside given limits. Besides, industrial exploitation of ecological systems (gathering crops, harvesting resources) leads to the need to optimize cropping, profitability, and so on. The possibilities of control, i.e., direct influence over the natural life of an ecosystem, are often bounded. In similar problems, it may make sense to pass from traditional formulations of control objectives (controlling and tracking) to "softer" setting such as partial stabilization, e.g., controlling the values of a certain function of the ecosystem state, which in absence of a controlling influence is preserved (is an invariant of the system's free motion). A recent work [71] proposes adaptive control algorithms for invariants of ecological networks defined by a multi-species Lotka–Volterra model. Note that Lotka–Volterra dynamics is closely related to game models of biological evolution of the species [72], in particular *replicator dynamics*. These models lay the foundations for a new rapidly growing field of *evolutionary game theory* [40, 73] that relates game theory and theory of complex networks and finds numerous applications in economics, biology, and "evolutionary algorithms" that are used in artificial intelligence.

Neural networks. A natural and very meaningful example of networks are neural networks composed of neural cells of a human being or an animal connected with electrical or biochemical interactions. The first models of such dynamical systems, known as **networks of pulse-coupled oscillators**, were studied in biological, physical, and mathematical literature long before the "boom" of network control theory (see, e.g., the well-known work [74] and bibliography therein). Pulse-coupled oscillators yield an example of a *hybrid* network control system where the dynamics of nodes is continuous, but interactions occur at discrete time moments that depend on the system state. In electrotechnics, this approach to control is known as "impulse modulation of the second kind," while modern control theory literature calls it "event-triggered control." For a number of important processes, e.g., *circadian* (daily) rhythms, there also exist continuous models [75] that reduce to the general form (1). At the same time, the above-mentioned models describe a rather narrow class of processes controlled by neural cells and reducing to generating stable periodic rhythms. The most interesting of those are neural networks that define the cognitive activity of the brain. Dynamics of these networks is so complex, and possibilities for control are so weak, that at first glance mathematical methods for network control are inapplicable to neural networks. However, one example of such an application is given by the field of neurofeedback connection that has lately developed in neural sciences and has achieved rapid growth [76]. It is based on using multidimensional signals read from an electric or magnetic magnetoencephalograph to measure activity states of the brain. For the control, one can show certain images on the computer screen based on how close the measured state is to the desired state. There are more and more examples where such schemes of neurophysiological studies have been successfully used to treat cerebral diseases.

Molecular systems and nanosystems. One well-known example of network structure in physics is a crystal lattice. The atoms oscillate around the nodes of a lattice under the interatomic forces that decay as the distance between the atoms grows. Although there are few ways to influence such systems, control problems under the influence of external forces and electromagnetic fields in certain cases may be solved, and their solutions may lead to creating new substances or materials with new, unusual properties. These problems have become especially relevant over the latest years when there appeared measurement and control devices (atom-force microscopes, computer controlled femtosecond lasers) that let us control the behavior of individual atoms and molecules [25]. We specifically mention a new field of control over *quantum* networks [77], where nodes carry quantum information (qubits), and links correspond to quantum entanglement between the states of qubits.

3. MATHEMATICAL MODELS OF NETWORK CONTROL

First of all, let us comment upon the difference between network systems control and multiconnected systems that have been traditional for the science of the XX century. A control plant (P) in a multivariable system also has several inputs and several outputs. However, studies of multivariable systems have traditionally assumed that measurement results of output values go to a central controlling device that implements the control algorithm for the object and produces values of controlling influences to further send them to P inputs. Under this assumption, the controller has access to all measured values of the outputs at the same time (Fig. 2, left). The theory of multivariable systems had been well developed already in the XX century [78–82].

Network control problems, on the other hand, do not have a single control plant and a single controller. There may be both multiple plants and multiple controllers, and they can be spatially distributed (Fig. 2, right). Moreover, each controller only has access to a part of the measured (output) variables, and time moments when information in the network is updated may differ in different parts of the network. Thus, plants and controllers are to be considered as interacting agents in a multiagent system, and instead of control algorithms we should speak of network protocols for interactions between subsystems (agents). This presents additional obstacles in setting and solving the problems but allows to include into consideration asynchronous and event-driven (event-triggered) interactions.

Next we consider formal problem settings for network control and main approaches to solving them. Consider a network S of the form (1) consisting of N interconnected subsystems (agents) S_i, $i = 1, \ldots, N$, each of which is defined by equations

$$\dot{x}_i = F_i(x_i, u_i) + \sum_{j=1}^{N} \alpha_{ij}\varphi_{ij}(x_i, x_j), \quad y_i = h_i(x_i, u_i), \quad i = 1, \ldots, N, \tag{2}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^l$ are the state, input (control), and output (measurement) vectors for agent S_i, vector function $F_i(x_i, u_i)$ and $h_i(x_i, u_i)$ define agent dynamics, vector functions $\varphi_{ij}(\cdot)$, $i = 1, \ldots, d$, $j = 1, \ldots, d$, define interactions between subsystems, and coefficients $\alpha_{ij} \in \mathbb{R}^1$ characterize the intensity of interactions. If all agents are identical, i.e., $F_i(x_i, u_i) = F(x_i, u_i)$, the network is called *homogenous*; otherwise, *heterogeneous*. There often arise networks where interactions between agents depend only on their disagreement (difference in their states): $\varphi_{ij}(x_i, x_j) = \varphi_{ij}(x_i - x_j)$. Such couplings, especially in case of linear functions φ_{ij}, are often called

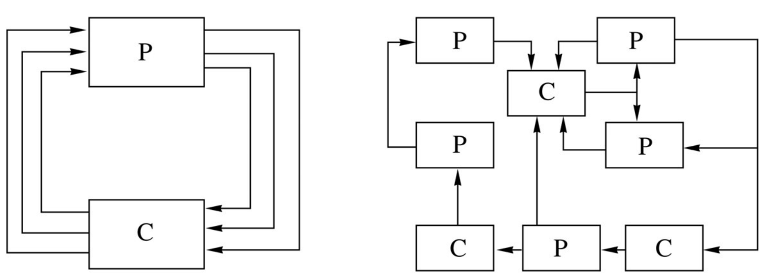

Fig. 2. A multivariable control system (left) and a network system (right). Blocks P and C denote respectively control plants and controllers.

diffusive coupling, and the network with such couplings is itself also sometimes called a diffusive network [83]. The most comprehensively studied type of network systems at present are linear network systems of the form

$$\dot{x}_i = A_i x_i + B_i u_i + \sum_{j=1}^{N} \alpha_{ij} A_{ij}(x_i - x_j), \quad y_i = C_i x_i + D_i u_i, \quad i = 1, \ldots, N, \tag{3}$$

where $A_i, B_i, C_i, D_i, A_{ij}$ are matrices of the corresponding dimensions.

Similarly, one can introduce classes of discrete time network models where derivatives are replaced with finite differences.

Properties of the network as a control object depend on the character of interactions between agents defined by functions $\varphi_{ij}(x_i, x_j)$ and coefficients α_{ij}. To define the structure of interactions, we introduce a directed graph (digraph) as follows: $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of vertices, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of arcs. For each $i = 1, \ldots, N$, vertex v_i is associated with agent S_i. We will assume that an arc (v_i, v_j) belongs to the set of arcs $\mathcal{E}$ if information flows from agent S_j to agent S_i. In terms of the model of network (2) it means that $\alpha_{ij} \neq 0$. We assume that the graph has no loops, i.e., $(v_i, v_i) \notin \mathcal{E}$ for all $i = 1, \ldots, N$. If $\alpha_{ij} \neq 0$ if and only if $\alpha_{ji} \neq 0$, the graph can be regarded as undirected. In the models considered below, all weights are positive:[3] $\alpha_{ij} > 0$. The resulting graph is called the *information graph*, or the network *topology*. As we show below, many fundamental properties of network systems are defined by their topology and can be naturally defined in the language of graph theory. Thus, graph theory plays a significant role in analysis and synthesis problems for network control systems.

4. OBJECTIVES OF NETWORK CONTROL: SYNCHRONIZATION, CONSENSUS, SWARMING

4.1. A General Definition of Synchronization

In network control problems, the control objective is the desired group (collective, cooperative) behavior of agents in the network. An important class of control objectives is given by providing the desired coordinated operation of the agents called *synchronization*. In particular, it can be the matching or approximation of state variables for two or more subsystems or a coordinated change in certain quantitative characteristics of subsystems. In some cases, synchronization arises due to natural properties of the system of interacting objects itself. One example of this is *frequency synchronization* for oscillating or rotating bodies (see below). This situation is called *self-synchronization*. In other cases, to coordinate the behavior of objects one has to introduce into the system additional interactions or influences. Then one says of *forced* or *controlled synchronization*. In these cases, synchronization means rendering the processes to behave synchronously.

The first versions of general definitions for periodic processes have been proposed in [84] (concurrence or multiplicity of mean frequencies of oscillatory or rotational motions) and in [85] (existence of an asymptotically stable invariant torus of dimension $n - m$, where m is the degree of synchronization). The works [84, 86] also note that synchronization may mean coincidence of the values of certain functionals of system coordinates. For instance, as such functionals one can consider the moments when coordinates turn to zero or reach extreme values. A number of works have been devoted to synchronization of oscillations in phase synchronization systems [87] and more general models of periodic systems [88]. Over the course of study of synchronization of chaotic processes,

[3] Formally speaking, a nonzero weight can always be regarded as positive by switching the sign of function φ_{ij}. At the same time, in many network systems (e.g., social [40]) couplings between nodes can be meaningfully divided into attractive and repulsive. These systems are more convenient to define with weighted graphs with variable signs (signed graphs).

a number of new versions for the notion of synchronization have appeared: coordinate (identical) synchronization [89, 90], generalized synchronization [91], phase synchronization [92], and so on.

A general definition of synchronization properties that encompasses both the case of self-synchronization and the case of controlled synchronization has been proposed in [93] and further developed in [26, 94, 95]. Below, similar to [25, 26], we give a general definition of synchronization that lets us both get many definitions known in literature as special cases and define various types of complex behavior in networks.

Consider k processes (objects) the state of each of which at time moment t is characterized by a certain vector $x^{(i)}(t)$, $i = 1, \ldots, k$, where t changes on the interval $0 \leqslant t < \infty$. We first assume that all vector functions $x^{(i)}(t)$ belong to the same functional space $\mathcal{X}$.

Suppose that we know a certain numerical characteristic of these processes defined by time-dependent mappings $C_t : \mathcal{X} \to \mathcal{C}$, where $\mathcal{C}$ is the set of possible values of C_t. The characteristic C_t is called the *synchronization parameter*, or *synchronization index*. It is important that the characteristic C_t is assumed to be the same for all objects or processes. The value of C_t may be a scalar, vector, matrix, or a function, e.g., the frequency spectrum of the process on an infinite or some finite, either fixed or sliding, time interval. In order to be able to compare the values of the characteristic for different processes, we introduce a set of time-independent vector functions $F_i : \mathcal{C} \to \mathbb{R}^m$, $i = 1, \ldots, k$, called *comparison functions.*

Definition. We say that *synchronization of processes* $x^{(i)}(t)$, $i = 1, \ldots, k$, *holds with respect to characteristic* C_t and comparison functions F_i if there exist real numbers (temporal or phase shifts) τ_i, $i = 1, \ldots, k$, such that for all $t \geqslant 0$ it holds that

$$F_1\left(C_{t+\tau_1}[x_1]\right) = \ldots = F_k\left(C_{t+\tau_k}[x_k]\right). \tag{4}$$

By *approximate* synchronization (ε-synchronization) we mean the case when relations (4) hold only approximately, up to ε:

$$\left|F_i\left(C_{t+\tau_i}[x_i]\right) - F_j\left(C_{t+\tau_j}[x_j]\right)\right| \leqslant \varepsilon \qquad \forall i, j, \quad t \geqslant 0, \tag{5}$$

and by *asymptotic* synchronization, the case when the accuracy up to which relations (4) hold disappears over time:

$$\lim_{t\to\infty}\left|F_i\left(C_{t+\tau_i}[x_i]\right) - F_j\left(C_{t+\tau_j}[x_j]\right)\right| = 0. \tag{6}$$

Here and in what follows we denote by $|X|$ the Euclidean norm (square of the sum of squares of all components) for a vector or matrix X, unless specified otherwise.

For a given averaging operator $\langle\cdot\rangle_t$ on the interval $0 \leqslant s \leqslant t$, one can introduce the notion of *synchronization on average* as the fact that for all $t \geqslant 0$ it holds that

$$\langle Q_s\rangle_t < \varepsilon, \tag{7}$$

where Q_s is some scalar function (measure of desynchronization) that characterizes its deviation from the synchronous mode. The averaging operator is often defined as an integral operator $\langle Q_s\rangle_t = \frac{1}{t}\int_0^t Q_s ds$, and the measure of desynchronization Q_s as the mean squared deviation from the synchronous mode:

$$Q_t = \sum_{i,j=1}^{k}\left|F_i\left(C_{t+\tau_i}[x_i]\right) - F_j\left(C_{t+\tau_j}[x_j]\right)\right|^2. \tag{8}$$

Introducing the measure of desynchronization is an important application of the formal definition. It gives a possibility to construct regular synthesis procedures for control algorithms over synchronization: definitions of controlling influences that create a synchronous mode in the system or change its characteristics. Such algorithms may be developed, for instance, based on the fast gradient method, see [25, 96].

Remark 1. Relations (4) are sometimes more convenient to write as $k-1$ equalities

$$F_i\left(C_{t+\tau_i}[x^{(i)}(t)]\right) - F_k\left(C_{t+\tau_k}[x^{(k)}(t)]\right) = 0, \quad i = 1, \ldots, k-1. \tag{9}$$

Remark 2. A more general notion is *multiple synchronization*, that corresponds to the case when relations (4) are replaced with

$$n_1 F_1\left(C_{t+\tau_1}[x^{(1)}(t)]\right) = \ldots = n_k F_k\left(C_{t+\tau_k}[x^{(k)}(t)]\right), \tag{10}$$

and equalities (9) transform into

$$F_i\left(C_{t+\tau_i}[x^{(i)}(t)]\right) = \frac{n_k}{n_i} F_k\left(C_{t+\tau_k}[x^{(k)}(t)]\right), \quad i = 1, \ldots, k-1, \tag{11}$$

where n_i are the *multiplicity coefficients* of the synchronization.

4.2. Types of Synchronization. Consensus

The above definition encompasses the main types of synchronous behavior of processes that occur in practice. Let us consider several examples.

Example 1 (frequency (Huygens) synchronization). This kind of synchronization is introduced for processes for which the notion of *frequency* ω_i is well defined, in particular for periodic (oscillatory or rotational) processes. The C_t characteristic in this case is the average frequency $C_t = \omega_t =< \dot{x} >_t$ over a period $0 \leqslant s \leqslant t$, and the synchronization condition is that

$$\omega_t = n_i \omega^*,$$

where n_i are integer numbers (synchronization multiplicities), and ω^* is the so-called *synchronous frequency*. Therefore, it is natural to introduce comparison functions as $F_i(\omega_t) = \omega_t / n_i$. For $n_i = 1$, $i = 1, \ldots, k$, we have simple (non-multiple) synchronization.

This version of synchronization can be extended to nonperiodic processes if one can correctly define average frequencies. One can also consider the "piecewise periodic" case, when the set of all time moments is partitioned into intervals $\Delta_q = [t_q, t_{q+1})$, $q = 1, 2, \ldots$, such that all motions $y_i(\cdot)$ are periodic on every interval Δ_q with frequencies $\omega_i(t)$ that are piecewise constant functions.

An extended version of Huygens synchronization arises if we replace the requirement that mean frequencies coincide exactly with a requirement that the spectra are coherent in the following sense. We introduce positive scaling functions for spectra $\alpha_i(\omega)$, $\beta_i(\omega)$ for every system Σ_i, $i = 1, \ldots, k$. The synchronization parameter C is defined as a function J_ω:

$$C_\omega(y_i(\cdot)) = \alpha_i(\omega) S_i(\beta_i(\omega)\omega), \tag{12}$$

where S_i is the spectral density of the output signal $y_i(t)$ which is assumed to be correctly defined. The comparison functions can be introduced by corresponding the synchronization parameter C with a set of values of C_ω for a given set of frequencies.

Example 2 (extremal synchronization). Extremal synchronization is when scalar processes $x^{(i)}(t)$ achieve their extreme values simultaneously or with a certain delay [93, 94]. The synchronization index in this case is $C_t = t_i^*(t)$, the time when the ith process reaches extremum on the interval $0 \leqslant s \leqslant t$. Time shifts τ_i can be the intervals between moments when the first extremum is achieved by the ith and the first processes. For vector processes, we can consider synchronization of the extrema of the corresponding scalar components of vectors $x^{(i)}(t)$ or certain scalar functions of $x^{(i)}(t)$. Such synchronization is important for a number of chemical and biological processes.

Example 3 (phase synchronization). Systems with phase synchronization are well known in radio technics and communication theory [87, 88]. However, traditional technical applications usually synchronize periodic processes whose frequencies are constant or periodic functions of time. In the 1990s, physicists sparked an interest to studying synchronization for chaotic processes, for which researchers introduced generalized definitions of phase and phase synchronization [92]. The most natural way to introduce the notion of phase for a chaotic process is to consider the process' behavior between moments when it intersects a certain surface (Poincare sections). Here the synchronization index will be the value of the phase φ_t of process $x(t)$ lying in the interval from 0 to 2π and defined as $C_t[x] = \varphi_t = 2\pi \frac{t-t_n}{t_{n+1}-t_n} + 2\pi n$, $t_n \leqslant t < t_{n+1}$, where t_n is the time of the nth intersection of the process trajectory with the Poincare section [92].

For $k = 2$, choosing $F_1(\varphi_t) = F_2(\varphi_t) = \varphi_t$ we get *in-phase* synchronization. If we define the comparison functions as $F_1(\varphi_t) = \varphi_t$, $F_2(\varphi_t) = \varphi_t + \pi$, we get *antiphase* synchronization.

A somewhat more general notion of synchronization results if we take as the value of the synchronization index the value $C_t = t_*(t)$, where $t_*(t)$ is the last moment of intersecting the surface that does not exceed the moment t [93]. This approach also encompasses the case when there is no physically meaningful phase since the process is too irregular. In particular, if we take as the Poincare section a surface defined by equating to zero the time derivative of a certain scalar function of the process, we get extremal synchronization (see above).

Example 4 (coordinate synchronization). Starting from mid-1980s, researchers began to use the definition of synchronization for interrelated subsystems as matching coordinates in their state vectors [89]. This definition has become especially popular after the publication of a work by Pecora and Carroll on control over the synchronization of chaotic systems [90] (this paper has been cited more than 6000 times). Obviously, coordinate synchronization also fits the general definition proposed above if we introduce synchronization index $C_t(x_i) = x_i(t)$, where $x_i(t)$ denotes the value of the state vector for the ith subsystem (agent) at time moment t, and comparison functions are identity functions: $F_i(x) = x$, $i = 1, \ldots, k$. A significant number of works have been devoted to studies of asymptotic coordinate synchronization, when the control objective is to satisfy relations

$$\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0, \quad i, j = 1, \ldots, k. \tag{13}$$

Example 5 (generalized (partial) coordinate synchronization). Coordinate synchronization from the previous example is often called *full*, or *identical*, which emphasizes that all phase coordinates for the subsystems must match exactly. Another important practical case is when only a part of the phase coordinates for subsystems, certain functions of phase coordinates $y_i = h(x_i)$, or outputs have to match. The corresponding notion was introduced in [91] and called *generalized synchronization*. Obviously, generalized synchronization fits the scheme above for $C_t(x_i) = x_i(t)$ and $F_i(x) = h(x)$, $i = 1, \ldots, k$.

Accordingly, asymptotic partial synchronization can serve as a control objective:

$$\lim_{t\to\infty} |y_i(t) - y_j(t)| = 0, \quad i, j = 1, \ldots, k. \tag{14}$$

Note that achieving partial synchronization may be related to the system having symmetries and invariant manifolds [83].

Example 6 (consensus). A special case of coordinate synchronization (full or partial) is *consensus* (with respect to the state or to the output). If, apart from satisfying (13), there exists a (general) *limit* of the states $\lim_{t\to\infty} x_j(t)$, we say that consensus has been reached among agents (or, speaking more precisely, asymptotic consensus with respect to the state). Similarly, *partial consensus* (or consensus with respect to the output) is defined as a strengthened version of condition (14) that presumes the existence of a general limit $\lim_{t\to\infty} y_j(t) \in \mathbb{R}^m$. We note, however, that for special cases consensus may also mean other types of behavior. For instance, for agents with double integrator dynamics $\ddot{y}_j(t) = u_j(t)$ consensus usually means the existence of a general limit for velocity vectors $\dot{y}_j$, and here trajectories of the agents $y_j(t)$ are synchronized in the sense of (14) [14, 27]. Further, consensus may mean that the states or outputs converge to the state or output of a specific agent, called the leader (leader-following consensus), or simply to some predefined reference trajectory (reference tracking consensus) [14, 27]. Besides, many works refer to consensus as a synonym of either full or partial coordinate synchronization.

The term "consensus" is motivated by two applications that arose at the same time in sociology and applied statistics. The first application is related to opinion and social power dynamics in social groups. One of the first network models of this sort, describing an iterative process of averaging opinions, was proposed by the social psychologist French in 1956 [97, 98]. On the other hand, operations research and applied statistics considered the problem of making coordinated decisions by a group of experts [99], which also turned out to admit a linear iterative algorithm [100]. Later the behavior of such systems was actively studied with the formalism of Markov chains; see, e.g., the survey [28]. Main results on reaching consensus will be considered below.

Example 7 (discrete synchronization). Sometimes one has to consider coordinate synchronization which is discrete in time, when exact matching of outputs holds only at some discrete set of time moments $\{t_q\}$, $q = 1, 2, \ldots$. In this case, the synchronization index $C[y_i(\cdot)]$ depends on the set of values of the outputs[4] of processes $y_i = h(x_i)$ at time moments t_q and can be defined as an infinite sequence

$$C[y_i(\cdot)] = \{y_i(t_1), y_i(t_2), \ldots\}.$$

A variation of discrete coordinate synchronization occurs when $C_q[y_i] = t_q$, where t_q is the time moment when some coordinates or outputs $y_i(t)$ either approach a given point or intersect a given surface. Another variation is the case when the value of t_q is defined as the time of reaching the qth local extremum of the signal. This version is a special case of extremal synchronization (see Example 2) and reduces to the previous one if we remember that the extremal condition is that the time derivative equals zero.

Naturally, one has to impose additional constraints that ensure that all introduced values are correctly defined. It suffices to require that each trajectory intersects the section an infinite number of times, and the intersection moments contain arbitrarily large $t \geqslant 0$. Interestingly, in this way one can construct a generalized definition of phase for a nonperiodic process which lets one consider phase synchronization (Example 3) as a special case of discrete synchronization.

Other examples. The above definition lets us formalize various properties of the processes that are intuitively desirable for synchronization by choosing a corresponding synchronization index and comparison functions. For instance, to define coordinate synchronization of oscillatory processes that occur synchronously but have different amplitudes of (different) oscillations, we can introduce a synchronization index with a normalizing factor:

$$C_t[x] = \frac{x(t)}{\max\limits_{0\leqslant s\leqslant t} |x(s)|}.$$

[4] In a special case one could have $y_i = x_i$.

If one of two processes with period T is superimposed with irregular noise, as the synchronization index we could use the sliding average of the process: $C_t[x] = \frac{1}{T}\int_{t-T}^{t} x(s)ds$.

Various combinations of the definitions introduced above are possible. For instance, one typical situation in synchronization problems for networks of electrical and mechanical systems is when we need asymptotic matching of angular velocities (rotation frequencies) of the rotors under an additional condition that the differences in rotation phases tend to constant values. Such an objective condition corresponds to transient stability of the power system, sometimes also called Willems stability.

4.3. Swarming

In solving optimization and control problems, researchers often turn to analogies in the behavior of biological and technical systems; this approach has led to the discipline of *bionics*. The bionic approach has led to whole new directions of study in applied mathematics: evolutionary programming, genetic algorithms, and so on. Numerical methods also reflect ideas coming from the attempts to construct models of collective, network behavior of organisms: particle swarm optimization, ant colony optimization algorithms, and so on.

In network control problems, the most comprehensively studied direction is flocking. Instead of "flock," researchers also sometimes use the terms herd, swarm, or school of fish. One convenient, and hence common, formulation of the notion of a "flock" has been given by a computer graphics researcher K. Reynolds in 1987 [101] in the form of three rules. These rules served as a foundation for the first computer model of flock behavior and have attracted a lot of attention since then. The rules are as follows:

1) cohesion: do not stray far from the neighbors;

2) separation: avoid collisions with the neighbors;

3) alignment: match the speed with the speed of the neighbors.

Modeling algorithms based on these rules have led to a quite realistic description of swarm behavior. The works [17, 18, 22] propose control algorithms for agents that let one initiate swarm behavior in the presence of obstacles. The most popular approach to constructing such algorithms is based on introducing the so-called *potential function* that penalizes both violations of rules 1–3 and approaching the obstacles. Note that potential functions can be interpreted as objective functions, and algorithms can be constructed based on fast gradient methods [96].

5. FUNDAMENTAL RESULTS: FAX–MURRAY THEOREM, REN–BEARD THEOREM, OLFATI-SABER'S THEOREM, CHEN'S THEOREM

Next we show several basic results in the field of network control that have already become classical to this day and represent cornerstones of the theoretical foundations of this direction of study.

Probably the best known publications on control over network systems are papers whose leading co-author was Richard Murray from the California Institute of Technology [2–4]. The number of citations both for paper [3] and for the paper of A.S. Morse with co-authors on a similar topic [102] has exceeded 3000, which sets a record for works on automation and control systems in the last 20 years. The number of citations of paper [2] and a subsequent survey [4] exceed 1000 which also supports the conclusion that network control is very important and highly relevant nowadays.

5.1. Stability and Synchronization in Linear Networks: The Fax–Murray Theorem

The work [2] considers stabilization problem for a formation of identical agents (carriages) with respect to a given reference motion. Suppose that the dynamics of deviations of the ith agent from the reference is defined by equations

$$\dot{x}_i = A_0 x_i + B_0 u_i, \quad y_i = C_0 x_i, \quad i = 1, \ldots, N, \tag{15}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^l$ are respectively state (deviation from reference motion), input (control), and output (measurement) vectors for the ith agent, A_0, B_0, C_0 are the matrices of the corresponding sizes. The control that serves as input for the ith agent is produced by the controller that receives as input the weighted sum of deviations of the agent state from the states of its neighbors:

$$z_i = -|N_i|^{-1} \sum_{j \in N_i} C_0(x_i - x_j), \quad i = 1, \ldots, N, \tag{16}$$

where N_i is the set of neighbors of the ith agent, $|N_i|$ is the number of elements in N_i. Each controller is also assumed to be a linear dynamical system, defined by state equations

$$\begin{aligned} \dot{v}_i &= A_R v_i + B_{1,R} y_i + B_{2,R} z_i, \\ u_i &= C_R v_i + D_{1,R} y_i + D_{2,R} z_i, \end{aligned} \qquad i = 1, \ldots, N, \tag{17}$$

where $v_i \in \mathbb{R}^s$.

Placing agents into N vertices of a graph and connecting the ith vertex with every element from N_i, we get the informational graph of the network system (which is, generally speaking, directed). To study the properties of system (15), (16) it is convenient to introduce the **Laplace matrix** of this graph $L = (L_{ij})_{i,j=1}^N$, where

$$L_{ij} = \begin{cases} 1, & \text{for } i = j \\ -|N_i|^{-1}, & \text{for } j \in N_i \\ 0 & \text{for } j \notin \{i\} \cup N_i. \end{cases} \tag{18}$$

The following theorem formulates stability and synchronization conditions in the above-described network (multiagent) system.

Theorem 1 (Fax, Murray [2]). *System* (15), (16) *asymptotically stable* ($x_j(t) \to 0$ *for* $t \to \infty$ *for all* j) *if and only if all systems of the following form*

$$\dot{x}_i = A_0 x_i + B_0 u_i, \quad y_i = C_0 x_i, \quad z_i = \lambda_i C_0 x_i, \quad i = 1, \ldots, N, \tag{19}$$

closed by the controller (17), *where* λ_i *are the eigenvalues matrices* L, *are asymptotically stable. If this condition is satisfied for all* ***nonzero*** λ_i, *and the graph is strongly connected,*[5] *then the asymptotically stable consensus set is* $\{(x_1, \ldots, x_N) : x_1 = \ldots = x_N\}$, *in other words, coordinate synchronization holds,* $|x_i(t) - x_j(t)| \xrightarrow[t \to \infty]{} 0$.

The Fax–Murray theorem shows that the stabilizability of a target set in a complex system of N objects whose state vector has dimension $(n+s)N$ is defined by the stability of N systems of lower order $(n+s)$. The resolvability of the stabilization problem nontrivially depends on the spectrum of the Laplace matrices. Note that eigenvalues λ_i may be complex if L is not a symmetric matrix, i.e., the graph of information connections is directed.

We note that the second statement of the above Theorem 1 on achieving synchronization was not in fact explicitly formulated in [2] as a theorem, although it was used in the examples. This theorem was subsequently rediscovered and generalized many times; see, e.g., [24]. In particular,

[5] A directed graph is called *strongly connected* if for every pair of vertices i, j there exists a directed path from vertex i to vertex j.

the strong connectivity condition in the second part can be replaced with the existence of a directed spanning tree similar to the consensus algorithms considered below, and instead of (16) one can consider output variables of a more general form

$$z_i = -\sum_{j=1}^{N} \alpha_{ij} C_0(x_i - x_j), \quad i = 1, \ldots, N, \tag{20}$$

where $\alpha_{ij} \geqslant 0$. Accordingly, the informational graph is defined by adjacency matrix (α_{ij}), and a regular Laplace matrix should be replaced by its equivalent for weighted graphs that we will introduce below in the part on consensus algorithms. Note that the Fax–Murray theorem provides tools to solve both analysis and synthesis problems. For instance, for a given informational graph it lets one synthesize the controller (17) that ensures synchronization of agent states (a similar problem has been considered in [24]). Using the "internal model principle" and its nonlinear generalizations, the synthesis problem can often be solved for nonuniform linear [103] and nonlinear [104] agents. Analyzing the proof in the work of Fax and Murray, we see that their theorem actually holds for a very wide class of multiagent systems and can be extended to systems with discrete time, systems with periodic coefficients, systems with delays, and other infinite-dimensional models. The corresponding special cases that consider agents and controllers of a special form, are often published with no references to the original general result.

We should note that a special case of Theorem 1 related to scalar agents had been obtained much earlier by B.T. Polyak and Ya.Z. Tsypkin [105] and was called the stability criterion for *uniform systems*. The work [105] has studied a system that can be viewed as a network of agents of the form

$$f\left(\frac{d}{dt}\right) y_i(t) = u_i(t), \quad u_i(t) = \sum_{j=1}^{N} a_{ij} y_j(t), \quad i = 1, \ldots, N.$$

Here $f(p)$ is a polynomial or a rational transition function, and matrix $A = (a_{ij})$ is fixed (note that it does not have to have Laplacian structure). Stability of this system is equivalent to the fact that polynomials $f(p) - \lambda_i$ are Hurwitz for all eigenvalues of matrix A. Passing to the state space, this statement can be proven similarly to the Fax–Murray theorem [2]. The original work [105], however, proposed an elegant proof in the frequency domain. One late development of the ideas of Polyak and Tsypkin are frequency stability criteria for multidimensional network systems obtained in subsequent works of S. Hara; see, in particular, [106].

5.2. Master Stability Function and Networks of Nonlinear Agents

Ideas based on the above-formulated results of Fax and Murray also include the notion of a *master stability function* introduced by American physicists Pecora and Carroll in 1998 [107]. This notion lets us study the local stability of the target manifold for more general, nonlinear models of dynamics in uniform networks

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{N} l_{ij} \varphi(x_j), \quad i = 1, \ldots, N, \tag{21}$$

where the agents' own dynamics and connections between them are uniform. One usually assumes that the matrix L has Laplacian structure in the sense that $\sum_{j=1}^{N} l_{ij} = 0$ and $l_{ij} \geqslant 0$ for $i \neq j$. If at point x_0 it holds that $f(x_0) = 0$, $\varphi(x_0) = 0$, then, obviously, $\mathbf{x}_0 = (x_0, \ldots, x_0)$ is an equilibrium of system (21). We construct Jacobians of functions $f(x)$, $\varphi(x)$ at the equilibrium point: $Df(x_0)$, $D\varphi(x_0)$, and consider the differential system (master stability equation)

$$\dot{z}(t) = [Df(x_0) + \lambda D\varphi(x_0)] z(t). \tag{22}$$

If system (22) is exponentially stable (matrix $D[f(x_0) + \lambda\varphi(x_0)]$ is Hurwitz) for all eigenvalues of the Laplacian matrix L, then the equilibrium of $\mathbf{x_0}$ is asymptotically stable. If, on the other hand, the said condition holds for **nonzero** eigenvalues $\lambda \neq 0$, and matrix L corresponds to a strongly connected graph, then near the equilibrium position the trajectories are asymptotically synchronized: $|x_i(t) - x_j(t)| \to 0$.

For linear systems, the above result is virtually identical to the first part of the Fax–Murray theorem. However, physics and other natural sciences widely use the following more general result which was also formulated (without proof) by Pecora and Carroll in 1998. Let $s(t)$ be a solution of system $\dot{s} = f(s)$ that defines a synchronous mode $\bar{X}(t) = (s(t), \ldots, s(t))$ (obviously, due to property $\sum_{j=1}^N l_{ij} = 0$ it is a solution of system (21)). We compute the leading Lyapunov parameter $\alpha(\lambda)$ for the system linearized along $s(t)$,

$$\dot{z}(t) = [Df(s(t)) + \lambda D\varphi(s(t))]z(t), \tag{23}$$

as a function of the complex number λ and depict on the complex plane the set S of points λ for which $\alpha(\lambda) < 0$. The synchronous mode $\bar{X}(t)$ is asymptotically stable if all points in the Laplacian's spectrum belong to S, i.e., $\lambda_i \in S$, $i = 1, \ldots, N$. If the said condition holds only for nonzero eigenvalues, and the informational graph is strongly connected, then asymptotic synchronization holds near the solution $X(t)$. Thus, to test asymptotic stability (synchronization) for a system of order $N \times n$ it suffices to check asymptotic stability of N (respectively, $N - 1$) systems of order n.

5.3. Consensus Criteria: Agents As an Integrator

The work [3] has also significantly influenced the development of network control. In this work, for the first time among journal works on automated control the authors systematically used the notion of consensus[6] and presented the basic notions and results of algebraic graph theory, laying the foundation of modern network control theory. They also considered the case of networks with variable topology (switching graphs of connections) which is important for control in dynamic conditions, when the connection between a pair of agents may be violated or repaired during the system's operation. The work [3] studied a network of agents in the form of an integrator

$$\dot{x}_i = u_i, \quad y_i = x_i, \quad i = 1, \ldots, N. \tag{24}$$

The problem is posed as achieving, with a consensus algorithm (*protocol*)

$$u_i = -\sum_{j=1}^N a_{ij}(x_i - x_j), \tag{25}$$

the control objective

$$\lim_{t\to\infty} x_i(t) = x_*, \quad i = 1, \ldots, N \tag{26}$$

for some x_*. Here matrix (a_{ij}) defines a weighted graph: agent i is connected to agent j if and only if $a_{ij} > 0$. Here the gain a_{ij} is understood as the weight of the edge (i, j).

If, apart from (26), it additionally holds that

$$x_* = \frac{1}{N}\sum_{i=1}^N x_i(0), \tag{27}$$

we say that we have achieved *consensus on average*.

[6] A rather general convergence criterion for the consensus of protocols with discrete time was published much earlier in [108], but that work did not use the term "consensus" or methods of graph theory.

We remind that a graph is called *balanced* if incoming and outgoing degrees of vertices coincide for all graph vertices, i.e., if

$$\sum_i a_{ij} = \sum_i a_{ji} \qquad \forall j = 1, \ldots, N.$$

Theorem 2 (Olfati-Saber–Murray, [3]). *Suppose that the topology of a network with a directed graph of connections G is fixed, and the graph G is strongly connected. The consensus protocol* (25) *ensures that consensus on average is achieved if and only if the graph G is balanced.*

If the graph is not balanced, the consensus value can be computed as follows. We introduce a "weighted" Laplace matrix L similar to (18):

$$l_{ii} = \sum_{j \neq i} a_{ij}, \quad l_{ij} = -a_{ij} \quad \text{for } i \neq j. \tag{28}$$

It is easy to see that system (24), (25) can be rewritten as

$$\dot{x} = -Lx. \tag{29}$$

Under the strong connectivity assumption, matrix $L^\top$ has a unique eigenvector that corresponds to the zero eigenvalue [3]: $L^\top d = 0$. Using the theory of nonnegative matrices, one can show [3] that this vector is nonnegative. Since it is nonzero, we get that $\sum_{i=1}^N d_i > 0$. Due to (29) we find that $d^\top \dot{x} = 0$, which immediately implies the formula for the consensus value.

Corollary 1. *For a graph G satisfying Theorem 2, let d be the eigenvector of the transposed Laplace matrix $L^\top$ corresponding to the zero eigenvalue. Then $d_i \geqslant 0$, $\sum_{i=1}^N d_i > 0$, and the value x_* is*

$$x_* = \sum_{i=1}^N d_i x_i(0) \Big/ \sum_{i=1}^N d_i. \tag{30}$$

Thus, in this case also the "consensus" value x_* belongs to the convex hull of the set of initial conditions.

For undirected graphs, it suffices to require that the graph is connected, which means that $\operatorname{rank} L = N - 1$. In this case the function

$$\Phi_G(x) = x^T L x = \frac{1}{2} \sum_{i,j=1}^N a_{ij}(x_i - x_j)^2 \tag{31}$$

is nonnegative definite. Choosing the function

$$V(x) = \frac{1}{2}\|x\|^2 \tag{32}$$

as the Lyapunov function for system (24), (25), we get $\dot{V} = -\Phi_G(x) \leqslant 0$, which implies that we have achieved consensus (26).

In case of a balanced graph G the vector $\mathbf{1} = [1, \ldots, 1]^\top$ is an eigenvector of matrix L corresponding to the zero eigenvalue, i.e., $\mathbf{1}^T L = 0$, and it follows from (30) that we have achieved consensus on average.

Note that function $\Phi_G(x)$ can be considered as an objective function in the construction of control protocols. Indeed, for a network of integrator agents in case of an undirected graph of connections we can write the consensus protocol (25) as

$$u = -\nabla \Phi_G(x). \tag{33}$$

Thus, the search for consensus in system (24), (25) occurs according to the gradient algorithm of minimizing the function $\Phi_G(x)$.

It is interesting that (33) can also be interpreted as a fast gradient algorithm for control objects of the form (24) with objective function $Q(x)$ of the form (31). This becomes obvious if we write $\dot{Q}(x) = \nabla\Phi_G(x)^\top u$ and compute the gradient of this expression with respect to u. Generalizing this idea, one can extend the conclusions of the work [3] to networks whose agents' dynamics is nonlinear and defined by equations

$$\dot{x}_i = f(x_i) + g(x_i)u_i, \quad i = 1, \ldots, N. \tag{34}$$

A consensus protocol based on the fast gradient algorithm for network (34) has the form

$$u = -\gamma g^\top \nabla\Phi_G(x). \tag{35}$$

Results on convergence of such protocols for *passive* nonlinear agents can be found in [109, 110]. The works [29–34] consider cases of more general passivated agents. A number of results on the convergence of protocols (35) can be obtained for *incrementally* dissipative agents [111].

We note that results close to the results of [3] were first obtained in [102] for systems with discrete time and switching topology, where the graph $G = G(t)$ at every time moment is chosen from a predefined finite set of strongly connected graphs.

Computing the rate of change for the Lyapunov function (32), it is easy to show that function (32) decreases exponentially with decay rate

$$\kappa = \inf_{G(t)} \operatorname{Re}\lambda_2(G(t)). \tag{36}$$

In particular, for undirected graphs the rate of synchronization is characterized by *algebraic connectivity* of the graph of connections.

The above-mentioned results have been extended in [3] to protocols with delays

$$u_i(t) = -K \sum_{j\in N_i} a_{ij}(x_i(t-\tau_{ij}) - x_j(t-\tau_{ij})). \tag{37}$$

In case of identical delays ($\tau_{ij} \equiv \tau$) it has been shown that if the graph of connections G is fixed, undirected, and connected, then the protocol (37) ensures consensus if and only if either $0 < \tau < \pi\lambda_n/2$, where $\lambda_n = \max\lambda_k(L)$, or the hodograph of the frequency characteristic (Nyquist plot) for agent $r(s) = \exp(-\tau s)/s$ does not encircle the point $-1/\lambda_k$ for any $k = 2, \ldots, n$, where λ_k are the eigenvalues of the Laplace matrix L. Besides, for $\tau = \pi\lambda_n/2$ the network has a globally asymptotically stable periodic solution with frequency $\omega = \lambda_n$. A number of works (see [112–114] and references therein) have generalized these results to the cases of a variable graph and nonuniform delay variables.

The pioneering results of [102] were significantly developed and in a most elegant and finished form formulated in the paper by W. Ren and W.R. Beard [115]. The number of citations for this classical paper approaches 2000. In [115], the authors consider reaching consensus in a system of agents defined by integrators:

$$\dot{x}_i = u_i, \quad y_i = x_i, \quad i = 1, \ldots, N. \tag{38}$$

The graph of connections G is considered as a directed graph and may change over time: $G = G(t)$, and $G(t)$ is chosen from a finite set of graphs. To coordinate the states of agents, one uses the consensus protocol

$$u_i = -K \sum_{j\in N_i(t)} (x_i - x_j), \tag{39}$$

where $N_i(t)$ is the set of neighbors of the ith agent at time moment t and $K > 0$. The control objective is to reach asymptotic coordinate synchronization

$$\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0, \quad i, j = 1, \ldots, N, \tag{40}$$

which is called the consensus in [115]. In reality, one can show that consensus in the sense of a stronger condition (26) under the assumptions shown below can also be achieved, and, moreover, it is robust to the presence of delays in the communication system [116].

To formulate the result, we have to introduce the notion of a *spanning tree* in a graph (see the details in the Appendix). A subgraph D of digraph G is called an incoming spanning tree if there exists a vertex i_0 (the root) such that for every other vertex j there exists a directed path from i_0 to j consisting of arcs from D. We also remind that the union of graphs $G_1, \ldots, G_k$ that have the same set of vertices is the graph G with the same set of vertices and set of arcs which is the union of sets of arcs for the graphs $G_1, \ldots, G_k$.

The work [115] makes the assumption that the graph is piecewise constant and switches at discrete time moments t_i separated from each other with a positive delay (dwell time) $\tau_i = t_{i+1} - t_i \geqslant \tau > 0$. Under this assumption, the following consensus criterion holds.[7]

Theorem 3 (Ren–Beard). *The consensus protocol* (39) *ensures asymptotic consensus if there exists an infinite sequence of disjoint uniformly bounded in length time intervals* $\Sigma_j = [t_{i_j}, t_{i_{j+1}})$ *such that the union of graphs on every interval* Σ_j *has an incoming spanning tree.*

One can show [117, 118] that in case of undirected and balanced graphs the Ren–Beard condition can be significantly weakened, requiring only that the infinite union of graphs G_k has a spanning tree starting from any moment. If the union of graphs $G_1, \ldots, G_k$ after a certain finite the time moment does not have an incoming spanning tree, then the consensus cannot be reached.

Thus, the key condition for achieving consensus and synchronization is the existence of a spanning tree in the graph. It has been shown in [119], another foundational work (more than a thousand citations), that violating this condition means that the graph has two nonempty "isolated" subsets of vertices that do not include any one edge from the outside, and therefore the corresponding two groups of agents are completely independent of each other and cannot synchronize.

The meaning of this condition also becomes clear from the lemma proven in [115]; the lemma states that in order for a spanning tree to exist it is necessary and sufficient that the multiplicity of the zero eigenvalue of the graph's Laplace matrix equals one. This lemma is of primary importance because it relates the properties of a network with the properties of its graph of connections. If we accept the lemma, proof of the theorem for the case of a fixed topology becomes obvious. Indeed, if we introduce the state vector of the network as $X = [x_1, \ldots, x_N]^\top$, for a fixed topology the graph dynamics of the network can be defined with a vector differential equation

$$\dot{X} = -KLX. \tag{41}$$

If the multiplicity of the zero eigenvalue of matrix L equals one, the eigenspace of matrix L corresponding to the zero eigenvalue is one-dimensional. But all solutions of system (41) converge to this eigenspace since all other eigenvalues of matrix $(-KL)$ have negative real parts (see Appendix). Since the eigenvector corresponding to the zero eigenvalue of L has the form $\mathbf{1} = [1, \ldots, 1]^\top$, limit values of states of agents x_i^* will be the same: $x_i^* = x_j^*$, i.e., we will achieve consensus. One can

[7] In [115], an additional condition is also assumed: delays τ_i take values in an infinite set Υ that results from a *finite* set $\Theta \subset (0, \infty)$ by constructing all possible finite sums $\theta_1 + \ldots + \theta_k$, where $k \geqslant 1$ and $\theta_i \in \Theta$. A subsequent book [27] shows (Theorem 2.33) that the latter condition can be discarded. Besides, protocol (39) can be replaced with a more general protocol (25), where the weights $a_{ij}(t)$ are uniformly bounded.

show, moreover, that if the graph is balanced then the general limit value $x^* = x_i^*$ equals to the arithmetic average of the initial conditions (consensus on average).

It is important to note that the key lemma proven in [115], which relates the one-dimensionality of the zero eigenspace of the Laplace matrix with the existence of a spanning tree in the graph, is a special case of a much more general statement proven by Russian mathematicians R.P. Agaev and P.Yu. Chebotarev back in 2000 [120]. In a later formulation [120, 121], the Agaev–Chebotarev theorem says that the dimension of the zero eigenspace of the Laplace matrix for a digraph equals the forest dimension of the graph (the forest dimension of a graph is the minimal number of trees in its spanning forest; see Appendix). The Agaev–Chebotarev theorem lets one establish conditions for the so-called cluster synchronization under which the dimension of the limit subspace of the dynamical network exceeds one [122]. A sample problem where the graph does not have a spanning tree is the *containment control* problem [14], where the control objective is to hold a group of agents inside the convex hull of several completely independent leaders. The leaders are roots of the trees that form the spanning forest.

5.4. Second Order Criteria for Consensus and Synchronization

It is very hard to get conditions for achieving synchronization and consensus for networks of multidimensional agents. Despite the fact that the general Fax–Murray theorem formally implies synchronization conditions for agents of an arbitrary order, it is often nontrivial to analytically check simultaneous stability of several systems. The most successful in this regard has been an extension of the results obtained for networks of first order agents to networks of agents defined by second order differential equations. Such models correspond to the simplest mechanical and physical systems that have inertia. The best studied case is the case of agents whose dynamics is defined with a double integrator model corresponding to the motion of a material point.

Consider a network that consists of agents

$$\dot{x}_i = v_i, \quad \dot{v}_i = u_i, \quad i = 1, \ldots, N. \tag{42}$$

To coordinate the states of the agents, one uses a proportionally differential consensus protocol

$$u_i = -\alpha \sum_{j \in N_i} (x_i - x_j) - \beta \sum_{j \in N_i} (v_i - v_j), \tag{43}$$

where $\alpha > 0$, $\beta > 0$. Using the definition of the Laplace matrix $L = L_{ij}$, protocol (43) can be rewritten as

$$u_i = -\alpha \sum_{j=1}^{N} L_{ij} x_j - \beta \sum_{j=1}^{N} L_{ij} v_j. \tag{44}$$

It has been shown in [27, 123] that in order to achieve consensus in the network (42), (44) it is necessary and sufficient that the graph of connections has a spanning tree, and coefficients of protocol (44) satisfy

$$\frac{\beta^2}{\alpha} > \max_{2 \leqslant i \leqslant N} \frac{\operatorname{Im}(\lambda_i)^2}{\operatorname{Re}(\lambda_i)[\operatorname{Re}(\lambda_i)^2 + \operatorname{Im}(\lambda_i)^2]}, \tag{45}$$

where λ_i are the eigenvalues of the graph's Laplace matrix.

Besides, if consensus is achieved then

$$\begin{aligned}&\left\|v_i(t)-\sum_{j=1}^{N}\xi_j v_j(0)\right\|\to 0,\\&\left\|x_i(t)-\sum_{j=1}^{N}\xi_j x_j(0)-\sum_{j=1}^{N}\xi_j v_j(0)t\right\|\to 0\end{aligned}\tag{46}$$

for $t\to\infty$, where $\xi=\mathrm{col}\{\xi_1,\dots,\xi_N\}$ is the unique nonnegative left eigenvector of matrix L corresponding to eigenvalue 0 and satisfying equality $\xi^\top\mathbf{1}=1$.

Note that (45) holds for all $\alpha>0$, $\beta>0$ if the eigenvalues of L are real, e.g., if the graph of connections is undirected. The latter result remains true for a number of algorithms with variable topology [27].

The consensus criterion (45) has been extended in [123] to networks (42) with delaying connections defined by the following protocol:

$$u_i=-\alpha\sum_{j=1}^{N}L_{ij}x_j(t-\tau)-\beta\sum_{j=1}^{N}L_{ij}v_j(t-\tau),\tag{47}$$

where $\tau>0$ is the constant delay. Suppose that the network's graph of connections has a spanning tree, and (45) holds. Consensus in network (42), (44) is achieved if and only if the following relation holds:

$$\tau<\min_{2\leqslant i\leqslant N}\frac{\theta_{i1}}{\omega_{i1}},\tag{48}$$

where $0\leqslant\theta_{i1}<2\pi$ satisfies relations

$$\begin{aligned}\cos\theta_{i1}&=[\mathrm{Re}\,(\lambda_i)\alpha-\mathrm{Im}\,(\lambda_i)\omega_{i1}\beta]/\omega_{i1}^2,\\\sin\theta_{i1}&=[\mathrm{Re}\,(\lambda_i)\omega_{i1}\beta+\mathrm{Im}\,(\lambda_i)\alpha]/\omega_{i1}^2,\\\omega_{i1}^2&=(\|\lambda_i\|^2\beta^2+\sqrt{\|\lambda_i\|^4\beta^4+4\|\lambda_i\|^2\beta^2})/2,\end{aligned}$$

where λ_i are the eigenvalues of L.

Discretization (sampling) of controllers has an impact similar to the delaying effect. From the implementation point of view, discrete algorithms have their advantages: we have to store in memory the values of the measured output not on the entire delay interval but only at the previous sampling instant. Besides, we get spare time for computation between sampling instants. The work [124] studies the case when measurements of velocities are subject to discretization, and coordinates are measured without delay. The control protocol in this case has the form

$$u_i(t)=-\alpha\sum_{j=1}^{N}L_{ij}x_j(t)-\beta\sum_{j=1}^{N}L_{ij}v_j(t_k),\tag{49}$$

where t_k are the sampling instants, $T=t_{k+1}-t_k$ is the sampling interval. Protocol (49) corresponds to replacing the velocities in protocol (44) with finite differences $x(t)-x(t_k)$ for $0<t-t_k\leqslant T$.

The work [124] has established that if a digraph of network connections contains a spanning tree, then in order to achieve consensus it is necessary and sufficient that

$$0<\beta/\alpha<1,\quad f(\alpha,\beta,\lambda_i,T)>0,\quad i=2,\dots,N,\tag{50}$$

where

$$f(\alpha,\beta,\lambda_i,h)=\frac{(\beta/\alpha)^2}{1-\beta/\alpha}\left(\sin^2(d_iT)-\sinh^2(c_iT)\right)$$
$$\times\left(\cosh((c_iT)-\cos(d_iT))^2-4\sin^2(d_iT)\sinh^2(c_iT)\right),$$

$$c_i=\sqrt{|\alpha|(|\lambda_i|-\mathrm{sgn}(\alpha)\mathrm{Re}\,(\lambda_i))/2};$$
$$d_i=\sqrt{|\alpha|(|\lambda_i|+\mathrm{sgn}(\alpha)\mathrm{Re}\,(\lambda_i))/2}.$$

It is easy to see that relations (50) can be satisfied if we choose the parameters in such a way that the β/α ratio is close to one. For the case of a Laplace matrix with real spectrum, in particular for undirected graphs, the formulation can be simplified.

Corollary 2. *Suppose that the Laplace matrix of the graph of connections has real spectrum. Then to achieve consensus it is necessary and sufficient that*

$$0<\beta<\alpha,\quad \sqrt{\alpha\lambda_i}T\neq k\pi,\quad i=2,\ldots,N,\quad k=0,1,\ldots. \tag{51}$$

In particular, consensus is achieved if

$$0<T<\pi\sqrt{\alpha\lambda_N}. \tag{52}$$

Thus, for networks with a real spectrum of the Laplace matrix the consensus is achieved for a sufficiently small discretization step T. This seemingly natural property does not hold if the Laplace matrix has at least one complex eigenvalue, which has also been shown in [124].

A number of other consensus and synchronization criteria for second order agents can be found in the books [14, 27], and works of the first author [125–129]. The works [125–129] also consider more general synchronization criteria in networks of agents of arbitrary order under *nonlinear connections* that generalize the circular and Popov's absolute stability criteria to multiagent systems. These results can also be generalized to networks of nonlinear agents that possess the incremental dissipativity property [111].

5.5. Pinning Control

From the beginning of the 2000s, works began to appear on control over complex dynamical networks in case when a (possibly significant) part of dynamics equations of the network does not contain the control. This kind of control problems has been termed *pinning control*. The idea of pinning control came from biology. An important example here is the worm C. elegans which is widely used as a model organism in genetics, neurophysiology, development biology, and computational biology. Although it is only about 1 mm long, the worm has a ramified neural system with about 300 neurons and 5000 synaptic connection. For this worm, biologists have been able to answer the question of how many neurons one has to control to pass the excitation to any neuron in the organism. It turned out that to control this neural network one only needs 49 neurons, less than 17% of their total number [133].

Another example relates to schools of fish and swarms of bees that migrate in search of food. It turns out that a relatively small number of informed members (about 5 %) are able to influence the behavior of other group members and their ability to move towards the intended goal [134]. From the point of view of control theory, these 17 % of neurons and 5 % of bees can be considered as a group of controllable agents through which one can control the entire complex network. Obviously, such a control strategy is rather efficient and economical. On the other hand, natural questions arise: how many controllable nodes do we need to have in a network and which nodes should a controlling influence be applied in order to achieve the goal in the most efficient way? The first attempt to answer these questions was made back in 1997 [130]. Subsequently, a number of results have been obtained in the works of G. Chen and his colleagues [131, 132, 136–138]. Here we show one recent result for undirected network graphs, following [138]. For directed graphs, a similar result is shown in [136].

Consider a dynamical network defined by equations

$$
\begin{aligned}
\dot{x}_i &= f(x_i) + c\sum_{j=1}^{N} \alpha_{ij} H x_j + u_i, \quad i = 1, \ldots, l, \\
\dot{x}_i &= f(x_i) + c\sum_{j=1}^{N} \alpha_{ij} H x_j, \qquad\quad i = l+1, \ldots, N,
\end{aligned}
\tag{53}
$$

where l is the number of nodes subject to control. We assume for simplicity that agents are controlled by linear controllers

$$
u_k = -c\kappa_k H(x_k - s), \quad k = 1, \ldots, l, \tag{54}
$$

where $s = s(t)$ is the leader agent with model $\dot{s} = f(s)$, $\kappa_k > 0$ are the feedback coefficients. We denote $D = \mathrm{diag}\{\kappa_1, \ldots, \kappa_l, 0, \ldots, 0\}$. We call the function $\psi : R^n \times R \to R^n$ uniformly V-decreasing, where V is a square matrix, if there exists a number $\rho > 0$ such that for all $y, z \in R^n$ and $t \geqslant 0$ it holds that

$$
(z - y)^T V(\psi(z, t) - \psi(y, t)) \leqslant -\rho \|z - y\|^2. \tag{55}
$$

Suppose that there exist a diagonal positive definite matrix U symmetric to the positive definite matrix V and a square matrix T such that function $f(x) + Tx$ is uniformly V-decreasing, and matrix

$$
(U \otimes V)[\rho(A + D) \otimes H + I \otimes T]
$$

is symmetric and nonnegative definite. Then system (53), (54) achieves the control objective $x_i(t) - s(t) \to 0$ for $t \to \infty$.

6. CONCLUSION

The above-mentioned results show that control over network systems has already turned into an independent research direction that has a well-established mathematical formalism and important nontrivial results. In this survey, we have only touched upon the main results and problems that have defined the development of network control theory over the latest years. The utmost importance of the listed results is that they let us reduce the study of complex dynamical networks consisting of a large number of interacting agents to separate studies of the dynamics of an individual agent in the network and to the study of spectral properties of the graph of connections. This shows the characteristic features of network control problems.

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